

RIEMANN INTEGRABILITY VERSUS WEAK CONTINUITY

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ABSTRACT. In this paper we focus on the relation between Riemann integrability and weak continuity. A Banach space X is said to have the weak Lebesgue property if every Riemann integrable function from $[0, 1]$ into X is weakly continuous almost everywhere. We prove that the weak Lebesgue property is stable under ℓ_1 -sums and obtain new examples of Banach spaces with and without this property. Furthermore, we characterize Dunford-Pettis operators in terms of Riemann integrability and provide a quantitative result about the size of the set of τ -continuous non Riemann integrable functions, with τ a locally convex topology weaker than the norm topology.

1. INTRODUCTION

The study of the relation between Riemann integrability and continuity on Banach spaces started on 1927, when Graves showed in [13] the existence of a vector-valued Riemann integrable function not continuous almost everywhere (a.e. for short). Thus, the following problem arises:

Given a Banach space X , determine necessary and sufficient conditions for the Riemann integrability of a function $f : [0, 1] \rightarrow X$.

A Banach space X for which every Riemann integrable function $f : [0, 1] \rightarrow X$ is continuous a.e. is said to have the Lebesgue property (LP for short). All classical infinite-dimensional Banach spaces except ℓ_1 do not have the LP. For more details on this topic, we refer the reader to [12], [6], [24], [14] and [19].

Regarding weak continuity, Alexiewicz and Orlicz constructed in 1951 a Riemann integrable function which is not weakly continuous a.e. [2]. A Banach space X is said to have the weak Lebesgue property (WLP for short) if every Riemann integrable function $f : [0, 1] \rightarrow X$ is weakly continuous a.e. This property was introduced in [27]. Every Banach space with separable dual has the WLP and the example of [2] shows that $C([0, 1])$ does not have the WLP. Other spaces with the WLP, such as $L^1([0, 1])$, can be found in [5] and [28]. In this paper we focus on the relation between Riemann integrability and weak continuity. In Section 2 we present new results on the WLP. In particular, we prove that the James tree space JT does not have the WLP (Theorem 2.3) and we study when $\ell_p(\Gamma)$ and $c_0(\Gamma)$ have the WLP in the nonseparable case (Theorem 2.8). Moreover, we prove that the WLP is stable

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under ℓ_1 -sums (Theorem 2.13) and we apply this result to obtain that $\mathcal{C}(K)^*$ has the WLP for every compact space K in the class MS (Corollary 2.16).

Alexiewicz and Orlicz also provided in their paper an example of a weakly continuous non Riemann integrable function. V. Kadets proved in [15] that a Banach space X has the Schur property if and only if every weakly continuous function $f : [0, 1] \rightarrow X$ is Riemann integrable. Wang and Yang extended this result in [29] to arbitrary locally convex topologies weaker than the norm topology. In the last section of this paper we give an operator theoretic form of these results that, in particular, provides a positive answer to a question posed by Sofi in [26].

Terminology and Preliminaries. All Banach spaces are assumed to be real. In what follows, X^* denotes the dual of a Banach space X . The weak and weak* topologies of X and X^* will be denoted by ω and ω^* respectively. By an operator we mean a linear continuous mapping between Banach spaces. The Lebesgue measure in \mathbb{R} is denoted by μ . The interior of an interval I will be denoted by $\text{Int}(I)$. The density character $\text{dens}(T)$ of a topological space T is the minimal cardinality of a dense subset.

A partition of the interval $[a, b] \subset \mathbb{R}$ is a finite collection of non-overlapping closed subintervals covering $[a, b]$. A tagged partition of the interval $[a, b]$ is a partition $\{[t_{i-1}, t_i] : 1 \leq i \leq N\}$ of $[a, b]$ together with a set of points $\{s_i : 1 \leq i \leq N\}$ that satisfy $s_i \in (t_{i-1}, t_i)$ for each i . Let $\mathcal{P} = \{(s_i, [t_{i-1}, t_i]) : 1 \leq i \leq N\}$ be a tagged partition of $[a, b]$. For every function $f : [a, b] \rightarrow X$, we denote by $f(\mathcal{P})$ the Riemann sum $\sum_{i=1}^N (t_i - t_{i-1})f(s_i)$. The norm of \mathcal{P} is $\|\mathcal{P}\| := \max\{t_i - t_{i-1} : 1 \leq i \leq N\}$. We say that a function $f : [a, b] \rightarrow X$ is Riemann integrable, with integral $x \in X$, if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\|f(\mathcal{P}) - x\| < \varepsilon$ for all tagged partitions \mathcal{P} of $[a, b]$ with norm less than δ . In this case, we write $x = \int_a^b f(t)dt$.

The following criterion will be used for proving the existence of the Riemann integral of certain functions:

Theorem 1.1 ([12]). *Let $f : [0, 1] \rightarrow X$. The following statements are equivalent:*

- (1) *The function f is Riemann integrable.*
- (2) *For each $\varepsilon > 0$ there exists a partition \mathcal{P}_ε of $[0, 1]$ with $\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| < \varepsilon$ for all tagged partitions \mathcal{P}_1 and \mathcal{P}_2 of $[0, 1]$ that have the same intervals as \mathcal{P}_ε .*
- (3) *There is $x \in X$ such that for every $\varepsilon > 0$ there exists a partition \mathcal{P}_ε of $[0, 1]$ such that $\|f(\mathcal{P}) - x\| < \varepsilon$ whenever \mathcal{P} is a tagged partition of $[0, 1]$ with the same intervals as \mathcal{P}_ε .*

We will also be concerned about cardinality. Throughout this paper, \mathfrak{c} denotes the cardinality of the continuum and $\text{cov}(\mathcal{M})$ denotes the smallest cardinal such that there exist $\text{cov}(\mathcal{M})$ nowhere dense sets in $[0, 1]$ whose union is the interval $[0, 1]$. This cardinal coincides with the smallest cardinal such that there exist $\text{cov}(\mathcal{M})$ closed sets in $[0, 1]$ with Lebesgue measure zero whose union does not have Lebesgue measure zero (see [4, Theorem 2.6.14]).

A set $A \subset \mathbb{R}$ is said to be *strongly null* if for every sequence of positive reals $(\varepsilon_n)_{n=1}^\infty$ there exists a sequence of intervals $(I_n)_{n=1}^\infty$ such that $\mu(I_n) < \varepsilon_n$ for every $n \in \mathbb{N}$ and $A \subset \bigcup_{n \in \mathbb{N}} I_n$. We will be interested in the following result:

Theorem 1.2 ([22]). *A set $A \subset \mathbb{R}$ is strongly null if and only if for every closed set F with Lebesgue measure zero, the set $A + F = \{a + z : a \in A \text{ and } z \in F\}$ has Lebesgue measure zero.*

We will denote by $\text{non}(\mathcal{SN})$ the smallest cardinal of a non strongly null set. We have that $\aleph_1 \leq \text{cov}(\mathcal{M}) \leq \text{non}(\mathcal{SN}) \leq \mathfrak{c}$ and, under Martin's axiom, and therefore under the Continuum Hypothesis, $\text{non}(\mathcal{SN}) = \text{cov}(\mathcal{M}) = \mathfrak{c}$. Furthermore, if $\mathfrak{b} = \mathfrak{c}$ then $\text{non}(\mathcal{SN}) = \text{cov}(\mathcal{M})$. However, there exist models of ZFC satisfying $\text{cov}(\mathcal{M}) < \text{non}(\mathcal{SN})$. For further references and results on this subject we refer the reader to [3].

2. THE WEAK LEBESGUE PROPERTY

It is known that every Banach space with separable dual has the WLP [28]. Next theorem gives a generalization in terms of $\text{cov}(\mathcal{M})$.

Theorem 2.1. *Every Banach space X such that $\text{dens}(X^*) < \text{cov}(\mathcal{M})$ has the WLP.*

Proof. Let $D = \{x_i^*\}_{i \in I}$ be a dense subset in X^* with $|I| < \text{cov}(\mathcal{M})$ and take $f : [0, 1] \rightarrow X$ a Riemann integrable function. Then every function $x_i^* f$ is Riemann integrable. Let E_i be the set of points of discontinuity of $x_i^* f$ for every $i \in I$. Each E_i is a countable union of closed sets with measure zero, so $E = \bigcup_{i \in I} E_i$ has measure zero since $|I| < \text{cov}(\mathcal{M})$. We claim that f is weakly continuous at every point of E^c . Let $x^* \in X^*$ and let M be an upper bound for $\{\|f(t)\| : t \in [0, 1]\}$. Fix $\varepsilon > 0$ and $t \in E^c$. Then, there exists $x_i^* \in D$ such that $\|x_i^* - x^*\| < \frac{\varepsilon}{3M}$. Since $t \notin E_i$, there exists a neighbourhood U of t such that $|x_i^* f(t) - x_i^* f(t')| < \frac{\varepsilon}{3}$ for every $t' \in U$. Thus,

$$|x^* f(t) - x^* f(t')| \leq |x^* f(t) - x_i^* f(t)| + |x_i^* f(t) - x_i^* f(t')| + |x_i^* f(t') - x^* f(t')| < \varepsilon$$

for every $t' \in U$. \square

Corollary 2.2. *Every Banach space with separable dual has the WLP.*

The space ℓ_1 has the WLP because it has the LP. Since every asymptotic ℓ_1 space has the LP [19], the space Λ_T defined by Odell in [21] is a separable Banach space with nonseparable dual such that it does not contain an isomorphic copy of ℓ_1 but it has the WLP (it is asymptotic ℓ_1). On the other hand, the James tree space JT (see [1, Section 13.4]) is a separable Banach space with nonseparable dual such that it does not contain an isomorphic copy of ℓ_1 and it does not have the WLP:

Theorem 2.3. *The James tree space does not have the WLP.*

Proof. We represent the dyadic tree by

$$T = \{(n, k) : n = 0, 1, 2, \dots \text{ and } k = 1, 2, \dots, 2^n\}.$$

A node $(n, k) \in T$ has two immediate successors $(n+1, 2k-1)$ and $(n+1, 2k)$. Then, a segment of T is a finite sequence $\{p_1, \dots, p_m\}$ such that p_{j+1} is an immediate successor of p_j for every $j = 1, 2, \dots, m-1$. The James tree space JT is the completion of $c_{00}(T)$ with the norm

$$\|x\| = \sup \sqrt{\sum_{j=1}^l \left(\sum_{(n,k) \in S_j} x(n,k) \right)^2} < \infty,$$

where the supremum is taken over all $l \in \mathbb{N}$ and all sets of pairwise disjoint segments S_1, S_2, \dots, S_l . Let $\{e_{(n,k)}\}_{(n,k) \in T}$ be the canonical basis of JT , i.e. $e_{(n,k)}$ is the characteristic function of $(n,k) \in T$. Define $f : [0, 1] \rightarrow JT$ as follows:

$$f(t) = \begin{cases} e_{(n-1,k)} & \text{if } t = \frac{2k-1}{2^n} \text{ with } n \in \mathbb{N} \text{ and } k = 1, 2, \dots, 2^{n-1} \\ 0 & \text{in any other case.} \end{cases}$$

We claim that f is Riemann integrable. Fix $N \in \mathbb{N}$ and let $\{I_1, I_2, \dots, I_{2^N-1}\}$ be a family of closed disjoint intervals of $[0, 1]$ with

$$\sum_{1 \leq n \leq 2^N-1} \mu(I_n) \leq \frac{1}{2^N} \text{ and } \frac{n}{2^N} \in \text{Int}(I_n) \text{ for each } 1 \leq n \leq 2^N-1.$$

Let J_1, J_2, \dots, J_{2^N} be the closed disjoint intervals of $[0, 1]$ determined by

$$[0, 1] \setminus \bigcup_{1 \leq n \leq 2^N-1} \text{Int}(I_n).$$

Then, $\mu(J_n) \leq \frac{1}{2^N}$ and $\|\sum_{n=1}^{2^N} a_n f(t_n)\| \leq \sqrt{\sum_{n=1}^{2^N} a_n^2}$ for every $a_n \in \mathbb{R}$ and every $t_n \in J_n$ due to the definition of the norm in JT . Thus, any tagged partition \mathcal{P}_N with intervals $J_1, I_1, J_2, \dots, I_{2^N-1}, J_{2^N}$ and points $t_1, t'_1, t_2, \dots, t'_{2^N-1}, t_{2^N}$ satisfies

$$\begin{aligned} \|f(\mathcal{P}_N)\| &\leq \left\| \sum_{n=1}^{2^N} \mu(J_n) f(t_{2n-1}) \right\| + \sum_{n=1}^{2^N-1} \mu(I_n) \leq \\ &\leq \sqrt{\sum_{n=1}^{2^N} \mu(J_n)^2} + \frac{1}{2^N} \leq \sqrt{\sum_{n=1}^{2^N} \frac{1}{2^{2N}}} + \frac{1}{2^N} \leq \frac{2}{\sqrt{2^N}}. \end{aligned}$$

Hence, $\|f(\mathcal{P}_N)\| \xrightarrow{N \rightarrow \infty} 0$ and f is Riemann integrable with integral zero.

We show that f is not weakly continuous at any irrational point $t \in [0, 1]$. Fix a irrational point $t \in [0, 1]$. There exists a sequence of dyadic points $\left(\frac{2k_j-1}{2^{n_j}}\right)_{j=1}^{\infty}$ converging to t with $(n_j-1, k_j)_{j=1}^{\infty}$ a sequence in T such that $(n_{j+1}-1, k_{j+1})$ is an immediate successor of (n_j-1, k_j) for every $j \in \mathbb{N}$. Then, $\sum_{j=1}^{\infty} e_{(n_j-1, k_j)}^*$ is a functional in JT^* , so the sequence $f\left(\frac{2k_j-1}{2^{n_j}}\right) = e_{(n_j-1, k_j)}$ is not weakly null and f is not weakly continuous at t . \square

Corollary 2.4 ([2]). $\mathcal{C}([0, 1])$ does not have the WLP.

Proof. Since every subspace of a Banach space with the WLP has the WLP and every separable Banach space is isometrically isomorphic to a subspace of $\mathcal{C}([0, 1])$, it follows from the previous theorem and the separability of JT that $\mathcal{C}([0, 1])$ does not have the WLP. \square

Corollary 2.5. Let K be a compact Hausdorff space.

- (1) If K is metrizable, then $\mathcal{C}(K)$ has the WLP if and only if K is countable.
- (2) If $\mathcal{C}(K)$ has the WLP then K is scattered. The converse is not true since $c_0(\mathfrak{c})$ does not have the WLP (Theorem 2.8) and it is isomorphic to a $\mathcal{C}(K)$ space with K scattered.

Proof. If K is a countable compact metric space, then $\mathcal{C}(K)^*$ is separable [10, Theorem 14.24], so $\mathcal{C}(K)$ has the WLP (Theorem 2.1). If K is an uncountable compact metric space, then $\mathcal{C}(K)$ is isomorphic to $\mathcal{C}([0, 1])$ [1, Theorem 4.4.8], so $\mathcal{C}(K)$ does not have the WLP (Corollary 2.4). Finally, if K is not scattered, then $\mathcal{C}(K)$ has a subspace isomorphic to $\mathcal{C}([0, 1])$ (see the proof of [10, Theorem 14.26]), so $\mathcal{C}(K)$ does not have the WLP. \square

Remark 2.6. Let $\{X_i\}_{i \in \Gamma}$ be a family of Banach spaces. Define $X := (\bigoplus_{i \in \Gamma} X_i)_{\ell_p}$ with $1 < p < \infty$ or $X := (\bigoplus_{i \in \Gamma} X_i)_{c_0}$. If $f : [0, 1] \rightarrow X$ is a bounded function, then its set of points of weak discontinuity is $E = \bigcup_{i \in \Gamma} E_i$, where each E_i is the set of points of weak discontinuity of f_i and f_i is the i 'th coordinate of f . Thus, the countable ℓ_p -sum or c_0 -sum of Banach spaces with the WLP has the WLP. We cannot extend this result to uncountable ℓ_p -sums or c_0 -sums even when $X_i = \mathbb{R}$ for every $i \in \Gamma$ (Theorem 2.8).

Now, we study the WLP for the spaces of the form $c_0(\kappa)$ and $\ell_p(\kappa)$ with κ a cardinal.

Theorem 2.7. For any cardinal κ and any $1 < p < \infty$, $c_0(\kappa)$ has the WLP if and only if $\ell_p(\kappa)$ has the WLP.

Proof. If $\ell_p(\kappa)$ does not have the WLP, then there exists a Riemann integrable function $f : [0, 1] \rightarrow \ell_p(\kappa)$ which is not weakly continuous a.e. If $I : \ell_p(\kappa) \rightarrow c_0(\kappa)$ is the canonical inclusion, then the function $I \circ f$ is weakly continuous at a point $t \in [0, 1]$ if and only if f is weakly continuous at t by Remark 2.6. Therefore, $I \circ f$ is not weakly continuous a.e. Since I is an operator, $I \circ f$ is also Riemann integrable. Thus, $c_0(\kappa)$ does not have the WLP.

To prove the other implication, suppose $c_0(\kappa)$ does not have the WLP. Then, there exists a Riemann integrable function $f : [0, 1] \rightarrow c_0(\kappa)$ which is not weakly continuous a.e. Let f_α be the α 'th coordinate of f for every $\alpha \in \kappa$ and E_α^n be the set of points where f_α has oscillation strictly bigger than $\frac{1}{n}$ for every $n \in \mathbb{N}$. Note that each E_α^n has Lebesgue measure zero. Since f is not weakly continuous a.e., $\bigcup_{\alpha < \kappa} (\bigcup_{n \in \mathbb{N}} E_\alpha^n)$ has not Lebesgue measure zero, so there exists $n \in \mathbb{N}$ such that $\bigcup_{\alpha < \kappa} E_\alpha^n$ has not Lebesgue measure zero.

Set $F_0 := E_0^n$ and $F_\alpha := E_\alpha^n \setminus (\bigcup_{\beta < \alpha} E_\beta^n)$ for every $\alpha \in \kappa \setminus \{0\}$. The sets F_α are pairwise disjoint. Let $\chi_{F_\alpha} : [0, 1] \rightarrow \{0, 1\}$ be the characteristic function of F_α for every $\alpha < \kappa$ and $g : [0, 1] \rightarrow c_0(\kappa)$ the function defined by the formula $g(t) = \sum_{\alpha < \kappa} \chi_{F_\alpha}(t) e_\alpha$ for every $t \in [0, 1]$, where $\{e_\alpha\}_{\alpha < \kappa}$ is the canonical basis of X .

Notice that g is not weakly continuous a.e. since each χ_{F_α} is not continuous at any point of F_α (because $\mu(F_\alpha) = 0$) and $\bigcup_{\alpha < \kappa} F_\alpha = \bigcup_{\alpha < \kappa} E_\alpha^n$ is not Lebesgue null. We claim that g is Riemann integrable. Let $\varepsilon > 0$. Since f is Riemann integrable, there exists a partition \mathcal{P}_ε of $[0, 1]$ such that $\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\| < \frac{\varepsilon}{n}$ for all tagged partitions \mathcal{P}_1 and \mathcal{P}_2 of $[0, 1]$ that have the same intervals as \mathcal{P}_ε . For every $\alpha < \kappa$ and any tagged partitions \mathcal{P}_1 and \mathcal{P}_2 of $[0, 1]$ that have the same intervals as \mathcal{P}_ε ,

$$|\chi_{F_\alpha}(\mathcal{P}_1) - \chi_{F_\alpha}(\mathcal{P}_2)| \leq \sum_{i=1}^N \mu(I_i) \leq n |f_\alpha(\mathcal{P}'_1) - f_\alpha(\mathcal{P}'_2)| \leq n \|f(\mathcal{P}'_1) - f(\mathcal{P}'_2)\| < \varepsilon$$

for suitable tagged partitions \mathcal{P}'_1 and \mathcal{P}'_2 of $[0, 1]$ with the same intervals as \mathcal{P}_ε , where I_1, I_2, \dots, I_N are the intervals of \mathcal{P}_ε whose interior has non-empty intersection with E_α^n .

Therefore, g is Riemann integrable. Let $h : [0, 1] \rightarrow \ell_p(\kappa)$ be the function given by the formula $h(t) = \sum_{\alpha < \kappa} \chi_{F_\alpha}(t) e_\alpha$. Since the sets F_α are pairwise disjoint, the function h is well-defined. Moreover, h is not weakly continuous a.e. because $I \circ h = g$. Set $F = \bigcup_{\alpha < \kappa} F_\alpha$ and $\phi : F \rightarrow \kappa$ such that $\phi(t) = \alpha$ if $t \in F_\alpha$. We claim that h is Riemann integrable with integral zero. Let $\varepsilon > 0$ and $\mathcal{P}_\varepsilon = \{I_1, I_2, \dots, I_M\}$ be a partition of $[0, 1]$ such that $\|g(\mathcal{P}')\| < \varepsilon$ for any tagged partition \mathcal{P}' of $[0, 1]$ with the same intervals as \mathcal{P}_ε . Notice that

$$(1) \quad \mu \left(\bigcup_{\text{Int}(I_i) \cap F_\alpha \neq \emptyset} I_i \right) < \varepsilon \text{ for every } \alpha < \kappa.$$

Thus, for any tagged partition $\mathcal{P} = \{(s_i, I_i)\}_{i=1}^M$ the following inequalities hold:

$$\begin{aligned} \|h(\mathcal{P})\| &= \left\| \sum_{s_i \in F} \mu(I_i) e_{\phi(s_i)} \right\| = \left\| \sum_{\alpha < \kappa} \mu \left(\bigcup_{\phi(s_i)=\alpha} I_i \right) e_\alpha \right\| = \\ &= \left(\sum_{\alpha < \kappa} \mu \left(\bigcup_{\phi(s_i)=\alpha} I_i \right)^p \right)^{\frac{1}{p}} = \left(\sum_{\alpha < \kappa} \mu \left(\bigcup_{\phi(s_i)=\alpha} I_i \right)^{p-1} \mu \left(\bigcup_{\phi(s_i)=\alpha} I_i \right) \right)^{\frac{1}{p}} \leq \\ &\stackrel{(1)}{\leq} \varepsilon^{\frac{p-1}{p}} \left(\sum_{\alpha < \kappa} \mu \left(\bigcup_{\phi(s_i)=\alpha} I_i \right) \right)^{\frac{1}{p}} \leq \varepsilon^{\frac{p-1}{p}} \end{aligned}$$

Therefore, h is Riemann integrable with Riemann integral zero. \square

The LP is separably determined [24]. Nevertheless, it follows from the following theorem that the WLP is not separably determined, since every separable infinite-dimensional subspace of $\ell_2(\kappa)$ is isomorphic to ℓ_2 (which has separable dual).

Theorem 2.8. *Let κ be a cardinal and $X = c_0(\kappa)$ or $X = \ell_p(\kappa)$ with $1 < p < \infty$.*

- (1) *If $\kappa < \text{cov}(\mathcal{M})$ then X has the WLP.*
- (2) *If $\kappa \geq \text{non}(\mathcal{SN})$ then X does not have the WLP.*

Proof. It is enough to prove the result when $X = c_0(\kappa)$ due to Theorem 2.7. Since $c_0(\kappa)^* = \ell_1(\kappa)$ has density character κ , it follows from Theorem 2.1 that $c_0(\kappa)$ has the WLP if $\kappa < \text{cov}(\mathcal{M})$.

Suppose $\text{non}(\mathcal{SN}) \leq \kappa \leq \mathfrak{c}$. Due to Theorem 1.2, there exist a closed Lebesgue null set F and a set $E = \{x_\alpha\}_{\alpha < \kappa}$ in \mathbb{R} such that $E + F$ does not have Lebesgue measure zero. Without loss of generality, we may assume that $E, F \subset [0, 1]$ and $(E + F) \cap [0, 1]$ does not have Lebesgue measure zero. Set $F_0 := (x_0 + F) \cap [0, 1]$ and $F_\alpha := ((x_\alpha + F) \cap [0, 1]) \setminus \left(\bigcup_{\beta < \alpha} F_\beta \right)$ for every $0 < \alpha < \kappa$. Let $\chi_{F_\alpha} : [0, 1] \rightarrow \{0, 1\}$ be the characteristic function of F_α for every $\alpha < \kappa$ and $f : [0, 1] \rightarrow X$ the function defined by the formula $f(t) = \sum_{\alpha < \kappa} \chi_{F_\alpha}(t) e_\alpha$ for every $t \in [0, 1]$, where $\{e_\alpha\}_{\alpha < \kappa}$ is the canonical basis of $c_0(\kappa)$.

Since the sets F_α are pairwise disjoint, the function f is well-defined. Each χ_{F_α} is not continuous at F_α , since F_α cannot contain an interval of $[0, 1]$. Therefore, f is not weakly continuous a.e. because $\bigcup_{\alpha < \kappa} F_\alpha = (E + F) \cap [0, 1]$ does not have Lebesgue measure zero.

We claim that f is Riemann integrable. For every $\alpha < \kappa$ and every tagged partition $\mathcal{P} = \{(s_i, I_i)\}_{i=1}^N$ we have

$$\chi_{F_\alpha}(\mathcal{P}) = \sum_{i=1}^N \mu(I_i) \chi_{F_\alpha}(s_i) \leq \sum_{i=1}^N \mu(I_i - x_\alpha) \chi_F(s_i - x_\alpha) = \chi_F(\mathcal{P}')$$

for a suitable tagged partition \mathcal{P}' with $\|\mathcal{P}\| = \|\mathcal{P}'\|$. Since $F \subset [0, 1]$ is a closed Lebesgue measure zero set, the characteristic function χ_F is Riemann integrable due to Lebesgue's Theorem. Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\chi_F(\mathcal{P}) < \varepsilon$ for every tagged partition \mathcal{P} with $\|\mathcal{P}\| < \delta$. Therefore, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\chi_{F_\alpha}(\mathcal{P}) < \varepsilon$ for all tagged partitions \mathcal{P} with $\|\mathcal{P}\| < \delta$ and for every $\alpha < \kappa$. Thus, f is Riemann integrable since $\|f(\mathcal{P})\| = \sup_{\alpha < \kappa} \chi_{F_\alpha}(\mathcal{P}) < \varepsilon$ for every tagged partition \mathcal{P} of $[0, 1]$ with $\|\mathcal{P}\| < \delta$. \square

The facts that the countable ℓ_1 -sum of spaces with the WLP has the WLP (Theorem 2.11) and that $L^1(\lambda)$ has the WLP if $\text{dens}(L^1(\lambda)) < \text{cov}(\mathcal{M})$ (Theorem 2.12) will be a consequence of the following lemma.

Lemma 2.9. *Let $(\Omega, \Sigma, \lambda)$ be a probability space and $\mathfrak{P} = \{P_A : A \in \Sigma\}$ a family of operators on a Banach space X such that*

- (1) $P_A + P_{\Omega \setminus A} = P_\Omega = \text{id}_X$ for every $A \in \Sigma$;
- (2) $\|P_A(x)\| \leq \|x\|$ for every $x \in X$ and every $A \in \Sigma$;
- (3) $\|P_A(x)\| + \|P_B(x')\| \leq \max\{\|x+x'\|, \|x-x'\|\}$ for every $x, x' \in X$ whenever $A \cap B = \emptyset$;
- (4) $\lim_{\lambda(A) \rightarrow 0} \|P_A(x)\| = 0$ for every $x \in X$.

Let $f : [0, 1] \rightarrow X$ be a Riemann integrable function. Then there is a measurable set $E \subseteq [0, 1]$ with $\mu(E) = 1$ such that, for every sequence $(t_n)_{n=1}^\infty$ in $[0, 1]$ converging to some $t \in E$, the set $\{f(t_n) : n \in \mathbb{N}\}$ is \mathfrak{P} -uniformly integrable, in the sense that

$$\lim_{\lambda(A) \rightarrow 0} \sup_{n \in \mathbb{N}} \|P_A(f(t_n))\| = 0.$$

Proof. The proof is similar to that of [5, Lemma 2.3] and [28, Lemma 3]. Fix $\beta > 0$ and denote by E_β the set of points $t \in [0, 1]$ such that for every $\delta > 0$ there exist $t' \in [0, 1]$ with $|t' - t| < \delta$ and a set $A \in \Sigma$ with $\lambda(A) < \delta$ such that

$$\|P_A(f(t) - f(t'))\| > \beta.$$

Let μ^* be the Lebesgue outer measure in $[0, 1]$. We show that $\mu^*(E_\beta) = 0$ with a proof by contradiction. Suppose $\mu^*(E_\beta) > 0$. Since f is Riemann integrable, we can choose a partition $\mathcal{P} = \{J_1, \dots, J_m\}$ of $[0, 1]$ such that

$$(2) \quad \left\| \sum_{j=1}^m \mu(J_j)(f(\xi_j) - f(\xi'_j)) \right\| < \beta \mu^*(E_\beta)$$

for all choices $\xi_j, \xi'_j \in J_j, 1 \leq j \leq m$. Let $S = \{j \in \{1, \dots, m\} : I_j \cap E_\beta \neq \emptyset\}$, where $I_j = \text{Int}(J_j)$ for each $j = 1, \dots, m$. Thus,

$$(3) \quad \sum_{j \in S} \mu^*(I_j \cap E_\beta) = \mu^*(E_\beta).$$

It is not restrictive to suppose $S = \{1, \dots, n\}$ for some $1 \leq n \leq m$.

Because of the definition of E_β and I_1 , there exist points $t_1 \in I_1 \cap E_\beta$ and $t'_1 \in I_1$ such that $\|f(t_1) - f(t'_1)\| \geq \|P_A(f(t_1) - f(t'_1))\| > \beta$ for some $A \in \Sigma$, hence $\|\mu(I_1)(f(t_1) - f(t'_1))\| > \beta\mu(I_1)$.

Fix $1 \leq k < n$ and assume that we have already chosen points $t_j, t'_j \in I_j$ for all $1 \leq j \leq k$ with the property that

$$\left\| \sum_{j=1}^k \mu(I_j)(f(t_j) - f(t'_j)) \right\| > \beta \left(\sum_{j=1}^k \mu(I_j) \right).$$

Define $x := \sum_{j=1}^k \mu(I_j)(f(t_j) - f(t'_j)) \in X$ and

$$\alpha := \|x\| - \beta \left(\sum_{j=1}^k \mu(I_j) \right) > 0.$$

Due to (4), we can choose $\delta > 0$ such that $\|P_A(x)\| < \alpha$ whenever $A \in \Sigma$ satisfies $\lambda(A) < \delta$. Take $t_{k+1}, t'_{k+1} \in I_{k+1}$ and a set $A \in \Sigma$ with $\lambda(A) < \delta$ such that $\|P_A(f(t_{k+1}) - f(t'_{k+1}))\| > \beta$, so $y := \mu(I_{k+1})(f(t_{k+1}) - f(t'_{k+1}))$ satisfies

$$\|P_A(y)\| > \beta\mu(I_{k+1}).$$

By the choice of A , (1) and (3), we also have (interchanging the role of t_{k+1} and t'_{k+1} if necessary)

$$\begin{aligned} \left\| \sum_{j=1}^{k+1} \mu(I_j)(f(t_j) - f(t'_j)) \right\| &\geq \|P_A(y)\| + \|P_{A^c}(x)\| \geq \|P_A(y)\| + \|x\| - \|P_A(x)\| > \\ &> \beta\mu(I_{k+1}) + \alpha + \beta \sum_{j=1}^k \mu(I_j) - \|P_A(x)\| > \beta \sum_{j=1}^{k+1} \mu(I_j). \end{aligned}$$

Thus, there exist $t_j, t'_j \in I_j$ for all $1 \leq j \leq n$ such that

$$\left\| \sum_{j=1}^n \mu(I_j)(f(t_j) - f(t'_j)) \right\| > \beta \left(\sum_{j=1}^n \mu(I_j) \right) \stackrel{(3)}{\geq} \beta\mu^*(E_\beta),$$

which contradicts the inequality (2). So we can conclude that $\mu^*(E_\beta) = 0$.

Therefore, $E := [0, 1] \setminus \bigcup_{n \in \mathbb{N}} E_{\frac{1}{n}}$ is measurable with $\mu(E) = 1$. Fix $t \in E$ and $m \in \mathbb{N}$. Since $t \notin E_{\frac{1}{m}}$, there exists $\delta_m > 0$ such that for every $t' \in [0, 1]$ with $|t' - t| < \delta_m$ and every set $A \in \Sigma$ with $\lambda(A) < \delta_m$,

$$\|P_A(f(t) - f(t'))\| \leq \frac{1}{m}.$$

Thus, for every $m \in \mathbb{N}$, every sequence $(t_n)_{n=1}^\infty$ converging to t and every $A \in \Sigma$ with $\lambda(A) < \delta_m$,

$$\|P_A(f(t_n))\| \leq \|P_A(f(t))\| + \frac{1}{m} \text{ for } n \text{ big enough depending only on } m.$$

Now the conclusion follows from (4). \square

Let $\{X_i\}_{i \in \Gamma}$ be a family of Banach spaces. We denote by $\pi_j : (\bigoplus_{i \in \Gamma} X_i) \rightarrow X_j$ the canonical projection onto X_j for each $j \in \Gamma$.

We will need the following property of ℓ_1 -sums and the space $L_1(\lambda)$ for Theorems 2.11 and 2.12:

Lemma 2.10. *Let $(\Omega, \Sigma, \lambda)$ be a probability space and $\{X_i\}_{i \in \Gamma}$ a family of Banach spaces. Then:*

- (1) $\max\{\|x + y\|, \|x - y\|\} \geq \sum_{i \in A} \|\pi_i(x)\| + \sum_{i \in B} \|\pi_i(y)\|$ for every vectors $x, y \in (\bigoplus_{i \in \Gamma} X_i)_{\ell_1}$ and any disjoint sets $A, B \subset \Gamma$.
- (2) $\max\{\|f + g\|, \|f - g\|\} \geq \int_A |f| d\lambda + \int_B |g| d\lambda$ for any $f, g \in L_1(\lambda)$ and any disjoint sets $A, B \in \Sigma$.

Proof. The second part is essentially Lemma 2 of [28]. The proof of the first part is analogous and we include it for the sake of completeness. Let $x, y \in (\bigoplus_{i \in \Gamma} X_i)_{\ell_1}$, $A, B \subset \Gamma$ be disjoint sets and $\varepsilon > 0$. Without loss of generality, we may assume that $A = \{a_n : n \in \mathbb{N}\}$ and $B = \{b_n : n \in \mathbb{N}\}$ are countable subsets. Consider the functionals $x^*, y^* \in (\bigoplus_{i \in \Gamma} X_i)_{\ell_1}^* = (\bigoplus_{i \in \Gamma} X_i^*)_{\ell_\infty}$ defined by $x^*(u) = \sum_{i \in A} x_i^*(\pi_i(u))$ and $y^*(u) = \sum_{i \in B} y_i^*(\pi_i(u))$ for every $u \in (\bigoplus_{i \in \Gamma} X_i)_{\ell_1}$, where each $x_i^*, y_i^* \in X_i^*$ satisfies $\|x_i^*\| = \|y_i^*\| = 1$, $x_i^*(\pi_i(x)) = \|\pi_i(x)\|$ if $i = a_n$ and $y_i^*(\pi_i(y)) = \|\pi_i(y)\|$ if $i = b_n$. Then, since A, B are disjoint, $\|x^* + y^*\| = \|x^* - y^*\| = 1$. Therefore,

$$\begin{aligned} \|x + y\| + \|x - y\| &\geq \langle x + y, x^* + y^* \rangle + \langle x - y, x^* - y^* \rangle = 2\langle x, x^* \rangle + 2\langle y, y^* \rangle = \\ &= 2\left(\sum_{i \in A} x_i^*(\pi_i(x)) + \sum_{i \in B} y_i^*(\pi_i(y))\right) = 2\left(\sum_{i \in A} \|\pi_i(x)\| + \sum_{i \in B} \|\pi_i(y)\|\right), \end{aligned}$$

so $\max\{\|x + y\|, \|x - y\|\} \geq \sum_{i \in A} \|\pi_i(x)\| + \sum_{i \in B} \|\pi_i(y)\|$. \square

Theorem 2.11. *Let $\{X_i\}_{i \in \mathbb{N}}$ be Banach spaces with the WLP. Then the space $X := (\bigoplus_{i \in \mathbb{N}} X_i)_{\ell_1}$ has the WLP.*

Proof. We are going to apply Lemma 2.9. Take $\Omega := \mathbb{N}$, $\Sigma := \mathcal{P}(\mathbb{N})$ the power set of \mathbb{N} , $\lambda(A) := \sum_{n \in A} 2^{-n}$ and $\mathfrak{P} = \{P_A : A \in \Sigma\}$ with

$$\pi_i(P_A(x)) = \begin{cases} \pi_i(x) & \text{if } i \in A \\ 0 & \text{if } i \notin A \end{cases}$$

for every $A \in \Sigma$ and every $x \in X$. Property (3) of Lemma 2.9 is Lemma 2.10(1) and property (4) holds because if $\lambda(A) < \frac{1}{2^n}$, then $A \subset \{n, n+1, \dots\}$, so

$$\|P_A(x)\| = \sum_{i \in A} \|\pi_i(x)\| \leq \sum_{i \geq n} \|\pi_i(x)\|$$

for every $x \in X$. Therefore, we can apply Lemma 2.9, so there exists a measurable set $E \subset [0, 1]$ with $\mu(E) = 1$ such that for every sequence $(t_n)_{n=1}^\infty$ in $[0, 1]$ converging to some $t \in E$ the set $\{f(t_n) : n \in \mathbb{N}\}$ is \mathfrak{P} -uniformly integrable. We can assume that, for each $i \in \mathbb{N}$, the map $t \mapsto \pi_i(f(t))$ is weakly continuous at each point of E because each X_i has the WLP.

It is a well known fact that a sequence $(x_n)_{n=1}^\infty$ in X converges weakly to $x \in X$ if and only if it satisfies the following two conditions:

- (i) $\pi_i(x_n) \rightarrow \pi_i(x)$ weakly in X_i for every $i \in \mathbb{N}$;
- (ii) for every $\varepsilon > 0$ there is a finite set $J \subseteq \mathbb{N}$ such that $\sup_{n \in \mathbb{N}} \|P_{\mathbb{N} \setminus J}(x_n)\| \leq \varepsilon$.

Since \mathfrak{P} -uniform integrability is equivalent to (ii), it follows that f is weakly continuous at each point of E . \square

A similar idea to that of Theorem 2.11 let us prove the following theorem, which improves [28, Theorem 5] and [5, Proposition 2.10].

Theorem 2.12. *Let $(\Omega, \Sigma, \lambda)$ be a probability space.*

- (1) If $\text{dens}(L^1(\lambda)) < \text{cov}(\mathcal{M})$ then $L^1(\lambda)$ has the WLP.
- (2) If $\text{dens}(L^1(\lambda)) \geq \text{non}(\mathcal{SN})$ then $L^1(\lambda)$ does not have the WLP.

Proof. Fix a Riemann integrable function $f: [0, 1] \rightarrow L^1(\lambda)$. Take $P_A(x) := x\chi_A$ for every $A \in \Sigma$ and every $x \in L^1(\lambda)$. The family of operators $\{P_A: A \in \Sigma\}$ fulfills the requirements of Lemma 2.9 (bear in mind Lemma 2.10). Then \mathfrak{P} -uniform integrability is the usual uniform integrability and therefore a set is bounded and \mathfrak{P} -uniformly integrable if and only if it is relatively weakly compact due to Dunford's Theorem (see [1, Theorem 5.2.9]). Lemma 2.9 ensures that there exist a measurable set $E \subset [0, 1]$ with $\mu(E) = 1$ such that for every sequence $(t_n)_{n=1}^\infty$ in $[0, 1]$ converging to some $t \in E$, the set $\{f(t_n): n \in \mathbb{N}\}$ is relatively weakly compact.

Let $\mathcal{C} \subset \Sigma$ be a dense family of λ -measurable sets, i.e. such that

$$\inf_{C \in \mathcal{C}} \lambda(A \triangle C) = 0 \text{ for every } A \in \Sigma.$$

Let $(h_n)_{n=1}^\infty$ be a relatively weakly compact sequence in $L^1(\lambda)$ and $h \in L^1(\lambda)$. Since \mathcal{C} is a dense family of λ -measurable sets, if $\int_C h_n d\mu \rightarrow \int_C h d\mu$ for every $C \in \mathcal{C}$, then $h = \omega\text{-lim } h_n$.

Suppose $\text{dens}(L^1(\lambda)) < \text{cov}(\mathcal{M})$. Then \mathcal{C} can be taken such that $|\mathcal{C}| < \text{cov}(\mathcal{M})$. Therefore, we can assume that, for each $C \in \mathcal{C}$, the Riemann integrable map $t \mapsto \int_C f(t) d\lambda$ is continuous at each point of E . Then, for every sequence $(t_n)_{n=1}^\infty$ in $[0, 1]$ converging to a point $t \in E$, we have $f(t) = \omega\text{-lim } f(t_n)$.

Now suppose $\nu = \text{dens}(L^1(\lambda)) \geq \text{non}(\mathcal{SN})$. Due to Maharam's Theorem (see [16, p. 127, Theorem 9]), $L^1(\lambda)$ contains an isometric copy of $L^1(\mu_\nu)$, where μ_ν is the usual product probability measure on $\{0, 1\}^\nu$. Since $L^1(\mu_\nu)$ contains an isomorphic copy of $\ell_2(\nu)$ (see [16, p. 128, Theorem 12]) and $\ell_2(\nu)$ does not have the WLP (Theorem 2.8), we conclude that $L^1(\lambda)$ does not have the WLP. \square

Theorem 2.11 can be extended to arbitrary ℓ_1 -sums:

Theorem 2.13. *The arbitrary ℓ_1 -sum of a family of Banach spaces with the WLP has the WLP.*

Proof. The proof uses some ideas of [18]. Let $f: [0, 1] \rightarrow X := (\bigoplus_{i \in \Gamma} X_i)_{\ell_1}$ be a Riemann integrable function, where $\{X_i\}_{i \in \Gamma}$ is a family of Banach spaces with the WLP. For each $J \subset \Gamma$, we denote by $P_J: X \rightarrow X$ the function defined by $\pi_i(P_J(x)) = \pi_i(x)$ if $i \in J$ and $\pi_i(P_J(x)) = 0$ in any other case. Let $(r_n)_{n=1}^\infty$ be an enumeration of the rational numbers in $[0, 1]$ and fix a countable set $L \subset \Gamma$ such that $P_L(f(r_n)) = f(r_n)$ for every $n \in \mathbb{N}$. Then, $f = (f - P_L f) + P_L f$. Since $P_L f$ is Riemann integrable and takes values in the space

$$X|_L := \{x \in X : \pi_i(x) = 0 \text{ for each } i \notin L\},$$

which is isomorphic to a countable ℓ_1 -sum of spaces with the WLP, by Theorem 2.11 $P_L f$ is weakly continuous almost everywhere.

Therefore, we can assume that $\int_0^1 f(t) dt = 0$ and that f is null over a dense set. Let

$$A_n^J := \{t \in [0, 1] : \|P_{J^c}(f(t))\| \geq \frac{1}{n}\}$$

for each $n \in \mathbb{N}$ and each subset $J \subset \Gamma$. If $J_1 \subset J_2 \subset \Gamma$, then $A_n^{J_2} \subset A_n^{J_1}$.

Claim: *For every $n \in \mathbb{N}$ there exists a countable set $J \subset \Gamma$ with $\mu(\overline{A_n^J}) = 0$.*

Suppose this is not the case. Then, there exist $n \in \mathbb{N}$ and $\delta > 0$ with $\mu(\overline{A_n^J}) > \delta$

for every countable subset $J \subset \Gamma$ (if for every $m \in \mathbb{N}$ we can take a countable set $J_m \subset \Gamma$ with $\mu(\overline{A_n^{J_m}}) < \frac{1}{m}$, then $J = \bigcup_{m \in \mathbb{N}} J_m$ verifies $\mu(\overline{A_n^J}) = 0$). Let $\mathcal{P} = \{I_1, I_2, \dots, I_N\}$ be a partition of $[0, 1]$ such that

$$(4) \quad \left\| \sum_{j=1}^N \mu(I_j)(f(\xi_j) - f(\xi'_j)) \right\| < \frac{\delta}{n} \text{ for all choices } \xi_j, \xi'_j \in I_j, 1 \leq j \leq N.$$

Let $J \subset \Gamma$ be a countable subset. Since $\sum_{j=1}^N \mu(I_j \cap \overline{A_n^J}) = \mu(\overline{A_n^J}) > \delta$ and f is null over a dense set, we can suppose that there exist $\xi_1 \in \text{Int}(I_1) \cap A_n^J$ and $\xi'_1 \in I_1$ such that $\|\mu(I_1)(f(\xi_1) - f(\xi'_1))\| \geq \frac{1}{n}\mu(I_1)$. Let $J_1 = \text{supp } f(\xi_1) \cup \text{supp } f(\xi'_1)$. By (4) we have $\mu(I_1) < \delta < \sum_{j=1}^N \mu(I_j \cap \overline{A_n^{J_1}})$ and so it is not restrictive to suppose $\text{Int}(I_2) \cap \overline{A_n^{J_1}} \neq \emptyset$. Thus, due to Lemma 2.10, we can choose $\xi_2, \xi'_2 \in I_2$ such that

$$\|\mu(I_1)(f(\xi_1) - f(\xi'_1)) + \mu(I_2)(f(\xi_2) - f(\xi'_2))\| \geq \frac{1}{n}(\mu(I_1) + \mu(I_2)).$$

Fix $1 \leq k < N$ and assume that we have already chosen points $\xi_j, \xi'_j \in I_j$ for all $1 \leq j \leq k$ with the property that

$$\left\| \sum_{j=1}^k \mu(I_j)(f(\xi_j) - f(\xi'_j)) \right\| \geq \frac{1}{n} \left(\sum_{j=1}^k \mu(I_j) \right).$$

Set $J_k := \bigcup_{j=1}^k \text{supp } f(\xi_j) \cup \text{supp } f(\xi'_j)$, which is countable. By (4) we have $\sum_{j=1}^k \mu(I_j) < \delta < \sum_{j=1}^N \mu(I_j \cap \overline{A_n^{J_k}})$, hence it is not restrictive to suppose that $\text{Int}(I_{k+1}) \cap \overline{A_n^{J_k}} \neq \emptyset$ and therefore that there exist points $\xi_{k+1}, \xi'_{k+1} \in I_{k+1}$ such that

$$\left\| \sum_{j=1}^{k+1} \mu(I_j)(f(\xi_j) - f(\xi'_j)) \right\| \geq \frac{1}{n} \left(\sum_{j=1}^{k+1} \mu(I_j) \right).$$

Since $\sum_{j=1}^N \mu(I_j) = 1 > \delta$, it follows that there exist $\xi_j, \xi'_j \in I_j$ for every $1 \leq j \leq N$ such that

$$\left\| \sum_{j=1}^N \mu(I_j)(f(\xi_j) - f(\xi'_j)) \right\| \geq \frac{\delta}{n}.$$

But this is a contradiction with (4). Therefore, the **Claim** is proved.

Thus, for every $n \in \mathbb{N}$ there exists a countable set J_n such that $\mu(\overline{A_n^{J_n}}) = 0$. Fix $J := \bigcup_{n \in \mathbb{N}} J_n$. Theorem 2.11 guarantees the existence of a set $F \subset [0, 1]$ of measure one such that $P_J(f)$ is weakly continuous at every point of F . Let $E = F \setminus \bigcup_{n \in \mathbb{N}} \overline{A_n^J}$. Then, $\mu(E) = 1$, $f = P_J(f) + P_{J^c}(f)$, $P_J(f)$ is weakly continuous at each point of E and $P_{J^c}(f)$ is norm continuous at each point of E (if $t_n \rightarrow t \in E$, then, for every $m \in \mathbb{N}$, $t_n \notin A_m^J$ for n big enough so $\|P_{J^c}(f)(t_n)\| < \frac{1}{m}$). \square

Corollary 2.14 ([24, 20]). $\ell_1(\kappa)$ has the LP for any cardinal κ .

Proof. Since $\ell_1(\kappa)$ has the Schur property, $\ell_1(\kappa)$ has the LP if and only if it has the WLP. Therefore, the conclusion follows from Theorem 2.13. \square

As an application of 2.13 we also obtain the following result:

Corollary 2.15. *Let K be a compact Hausdorff space. Then, $\mathcal{C}(K)^*$ has the WLP if $\text{dens}(L^1(\lambda)) < \text{cov}(\mathcal{M})$ for every regular Borel probability λ on K .*

Proof. For every compact Hausdorff space K , the Banach space $\mathcal{C}(K)^*$ is isometric to a ℓ_1 -sum of $L^1(\lambda)$ spaces, where each λ is a regular Borel probability measure on K (see the proof of [1, Proposition 4.3.8]). Thus, $\mathcal{C}(K)^*$ has the WLP if each space $L^1(\lambda)$ has the WLP, due to Theorem 2.13. Hence, the result follows from Theorem 2.12. \square

Corollary 2.16. *If K is a compact Hausdorff space in the class MS (i.e. $L^1(\lambda)$ is separable for every regular Borel probability on K), then $\mathcal{C}(K)^*$ has the WLP.*

Some classes of compact spaces in the class MS are metric compacta, Eberlein compacta, Radon-Nikodým compacta, Rosenthal compacta and scattered compacta. For more details on this class, we refer the reader to [8], [17] and [25].

The LP is a three-space property, i.e. if X is a Banach space and Y is a subspace of X such that Y and X/Y have the LP, then X has the LP [24, Proposition 1.19]. This result follows from Michael's Selection Theorem. However, as far as we are concerned, it is not known whether the WLP is a three-space property. We have a positive result in the following case:

Theorem 2.17. *Let X be a Banach space and Y a subspace of X . If Y is reflexive, $\text{dens}(Y) < \text{cov}(\mathcal{M})$ and X/Y has the WLP, then X has the WLP.*

Proof. Let $Q : X \rightarrow X/Y$ be the quotient operator and $\phi : X/Y \rightarrow X$ be a norm-norm continuous map such that $Q\phi = 1_{X/Y}$ given by Michael's Selection Theorem (see [10, Section 7.6]). Let $f : [0, 1] \rightarrow X$ be a Riemann integrable function. Then, since Qf is Riemann integrable and X/Y has the WLP, there exists a set $E \subset [0, 1]$ with $\mu(E) = 1$ such that Qf is weakly continuous at every point of E . Set

$$(5) \quad C = \{x \in X : \exists (t_n)_{n=1}^\infty \text{ converging to some } t \in E \text{ with } x = \omega\text{-}\lim f(t_n)\}.$$

First we are going to see that $\text{dens}(C) < \text{cov}(\mathcal{M})$. Let $x \in C$ and $(t_n)_{n=1}^\infty$ as in (5). Then $Qx = \omega\text{-}\lim Qf(t_n) = Qf(t)$. Therefore, $x = \phi(Qx) + (x - \phi(Qx))$ with $\phi(Qx) \in \phi(Qf(E))$ and $x - \phi(Qx) \in Y$. Notice that $\phi(Qf(E))$ is separable because of the ω -separability of $Qf(E)$ and Mazur's Lemma. Thus, $C \subset \phi(Qf(E)) + Y$ satisfies $\text{dens}(C) < \text{cov}(\mathcal{M})$.

Let $\{x_\alpha^*\}_{\alpha \in \Gamma} \subset X^*$ be a set separating points of C with $|\Gamma| < \text{cov}(\mathcal{M})$. Set $E_0 \subset E$ with $\mu(E_0) = 1$ such that $x_\alpha^* \circ f$ is continuous at every point of E_0 for every $\alpha \in \Gamma$. Notice that this can be done because the set of discontinuity points of each $x_\alpha^* \circ f$ is an F_σ Lebesgue null set and $|\Gamma| < \text{cov}(\mathcal{M})$. We claim that f is weakly continuous at each point of E_0 . Let $t \in E_0$ and $(t_n)_{n=1}^\infty$ be a sequence converging to t . Since $Qf(t) = \omega\text{-}\lim Qf(t_n)$, the set $\{Qf(t_n) : n \in \mathbb{N}\}$ is relatively weakly compact in X/Y . From the reflexivity of Y , it follows that Q is a Tauberian operator, so $\{f(t_n) : n \in \mathbb{N}\}$ is relatively weakly compact in X (see [11, Theorem 2.1.5 and Corollary 2.2.5]). Therefore, it is enough to prove the uniqueness of the limit of the subsequences of $(f(t_n))_{n=1}^\infty$. Let $x = \omega\text{-}\lim_k f(t_{n_k})$. Then, $x, f(t) \in C$ and $x_\alpha^*(x) = \lim_k x_\alpha^*(f(t_{n_k})) = x_\alpha^*(f(t))$ for every $\alpha \in \Gamma$, so $x = f(t)$. \square

3. WEAK CONTINUITY DOES NOT IMPLY INTEGRABILITY

It is not true that every weakly continuous function is Riemann integrable [2]. In fact, V. Kadets proved the following theorem:

Theorem 3.1 ([15]). *If X is a Banach space without the Schur property, then there is a weakly continuous function $f : [0, 1] \rightarrow X$ which is not Riemann integrable.*

The proof of the previous theorem together with Josefson-Nissenzweig Theorem (see [7, Chapter XII]) gives the following corollary:

Corollary 3.2. *Given an infinite-dimensional Banach space X , there always exists a weak* continuous function $f : [0, 1] \rightarrow X^*$ which is not Riemann integrable.*

In [29], Wang and Yang extend the previous result to a general locally convex topology weaker than the norm topology. In this section, we generalize these results in Theorem 3.4.

Following the terminology used in [9], we say that a subset M of a Banach space is spaceable if $M \cup \{0\}$ contains a closed infinite-dimensional subspace.

We start with the definitions of τ -Dunford-Pettis operator and the τ -Schur property, that coincide with the classical definitions of Dunford-Pettis or completely continuous operator and the Schur property when τ is the weak topology.

Definition 3.3. *Let X and Y be Banach spaces and τ a locally convex topology on X weaker than the norm topology. An operator $T : X \rightarrow Y$ is said to be τ -Dunford-Pettis (τ -DP for short) if it carries bounded τ -null sequences to norm null sequences. A Banach space X is said to have the τ -Schur property if the identity operator $I : X \rightarrow X$ is τ -DP.*

Theorem 3.4. *Let X and Y be Banach spaces and τ be a locally convex topology on X weaker than the norm topology. If $T : X \rightarrow Y$ is an operator which is not τ -DP, then the family of all bounded τ -continuous functions $f : [0, 1] \rightarrow X$ such that Tf is not Riemann integrable is spaceable in $\ell_\infty([0, 1], X)$, the space of all bounded functions from $[0, 1]$ to X with the supremum norm.*

Proof. The proof uses ideas from [15]. Since T is not τ -DP, we can take a bounded sequence $(x_n)_{n=1}^\infty$ that is τ -convergent to zero such that $\|Tx_n\| = 1$ for all $n \in \mathbb{N}$.

Let $K \subset [0, 1]$ be a copy of the Cantor set constructed by removing from $[0, 1]$ an open interval I_1^1 in the middle of $[0, 1]$ and removing open intervals $I_1^n, I_2^n, \dots, I_{2^n}^n$ from the middles of the remaining intervals in each step. Suppose that the removed intervals are so small that $\mu(K) > \frac{2}{3}$. Let $\mathcal{C}_a([0, 1])$ be the closed subspace of $\mathcal{C}([0, 1])$ consisting of all continuous functions $g : [0, 1] \rightarrow \mathbb{R}$ antisymmetric with respect to the axe $x = \frac{1}{2}$ and with $g(0) = g(1) = 0$. For every $g \in \mathcal{C}_a([0, 1])$ and every open interval $I = (a, b)$ in $[0, 1]$, we define the functions $g_I : [0, 1] \rightarrow \mathbb{R}$ and $f_g : [0, 1] \rightarrow X$ as follows

$$g_I(t) = \begin{cases} 0 & \text{if } t \notin (a, b), \\ g(\frac{t-a}{b-a}) & \text{if } t \in [a, b]. \end{cases}$$

$$f_g(t) = \begin{cases} 0 & \text{if } t \in K, \\ g_{I_k^n}(t)x_n & \text{if } t \in I_k^n. \end{cases}$$

The function $\phi : \mathcal{C}_a([0, 1]) \rightarrow \ell_\infty([0, 1], X)$ given by the formula $\phi(g) := f_g$ for every $g \in \mathcal{C}_a([0, 1])$ is a linear map and satisfies $\|\phi(g)\| = (\sup_n \|x_n\|)\|g\|$ for every

$g \in \mathcal{C}_a([0, 1])$. Therefore, ϕ is a multiple of an isometry. Thus, $V := \phi(\mathcal{C}_a([0, 1]))$ is an infinite-dimensional closed subspace of $\ell_\infty([0, 1], X)$.

We are going to check that each function $f_g \neq 0$ is τ -continuous but Tf_g is not Riemann integrable. Since g is continuous, $g(0) = g(1) = 0$ and $x_n \xrightarrow{\tau} 0$, f_g is τ -continuous. Suppose Tf_g is Riemann integrable. Then,

$$y^* \left(\int_0^1 Tf_g(t) dt \right) = \int_0^1 y^* Tf_g(t) dt = \sum_{k,n} y^*(Tx_n) \int_{I_k^n} g_{I_k^n}(t) dt = 0$$

for each $y^* \in Y^*$. The only possible value for the Riemann integral of Tf_g is 0 due to the above equality. Choose a partition $\mathcal{P} = \{J_1, J_2, \dots, J_N\}$ of $[0, 1]$. Let $A = \{j : 1 \leq j \leq N, \text{Int } J_j \cap K \neq \emptyset\}$. We can take $m \in \mathbb{N}$ such that if $j \in A$ then J_j contains some interval I_k^m . Hence, if $j \in A$, there is $t_j \in J_j$ such that $f_g(t_j) = \|g\|x_m$. If $j \notin A$, then we pick any $t_j \in \text{Int } J_j$. From the inequality $\sum_{j \in A} \mu(J_j) \geq \mu(K) > \frac{2}{3}$, we deduce

$$\begin{aligned} \left\| \sum_{j=1}^N \mu(J_j) Tf_g(t_j) \right\| &= \left\| \sum_{j \in A} \mu(J_j) Tf_g(t_j) + \sum_{j \notin A} \mu(J_j) Tf_g(t_j) \right\| \geq \\ &\geq \left\| \sum_{j \in A} \|g\| \mu(J_j) Tx_m \right\| - \left\| \sum_{j \notin A} \mu(J_j) Tf_g(t_j) \right\| > \frac{2}{3} \|g\| - \frac{1}{3} \sup_{t \in [0,1]} \|Tf_g(t)\| = \frac{1}{3} \|g\|. \end{aligned}$$

Then, Tf_g is Riemann integrable if and only if $g = 0$ if and only if $f_g = 0$. \square

The next corollary gives an affirmative answer to a question posed by Sofi in [26].

Corollary 3.5. *Given an infinite-dimensional Banach space X , the set of all weak* continuous functions $f : [0, 1] \rightarrow X^*$ which are not Riemann integrable is spaceable in $\ell_\infty([0, 1], X^*)$.*

Proof. X^* is not ω^* -Schur for any infinite-dimensional Banach space X due to the Josefson-Nissenzweig Theorem. Thus, the conclusion follows from Theorem 3.4. \square

Given a Banach space X , a function $f : [0, 1] \rightarrow X$ is said to be scalarly Riemann integrable if every composition x^*f with $x^* \in X^*$ is Riemann integrable.

We can also characterize Dunford-Pettis operators thanks to Theorem 3.4. The equivalence (1) \Leftrightarrow (3) in the following corollary was mentioned without proof in [23].

Corollary 3.6. *Let X and Y be Banach spaces and $T : X \rightarrow Y$ be an operator. The following statements are equivalent:*

- (1) *T is Dunford-Pettis.*
- (2) *Tf is Riemann integrable for every ω -continuous function $f : [0, 1] \rightarrow X$.*
- (3) *Tf is Riemann integrable for every scalarly Riemann integrable function $f : [0, 1] \rightarrow X$.*

Proof. (2) \Rightarrow (1) is a consequence of Theorem 3.4. Since every ω -continuous function $f : [0, 1] \rightarrow X$ is scalarly Riemann integrable, (3) implies (2). Therefore, it remains to prove (1) \Rightarrow (3). Suppose T is Dunford-Pettis and fix $(\mathcal{P}_n)_{n=1}^\infty$ a sequence of tagged partitions of $[0, 1]$ with $\|\mathcal{P}_n\| \xrightarrow{n} 0$. Let $f : [0, 1] \rightarrow X$ be a scalarly Riemann integrable function. Then, $x^*f(\mathcal{P}_n) \xrightarrow{n} \int_0^1 x^*f(t)dt$ for every

$x^* \in X^*$. Thus, $f(\mathcal{P}_n)$ is a ω -Cauchy sequence in X , so $Tf(\mathcal{P}_n)$ is norm convergent to some $y \in Y$. The limit y does not depend on the sequence of tagged partitions, since if $(\mathcal{P}'_n)_{n=1}^\infty$ is any other sequence of tagged partitions with $\|\mathcal{P}'_n\| \xrightarrow{n} 0$, then $f(\mathcal{P}_n) - f(\mathcal{P}'_n)$ is weakly null and this in turn implies that $\|Tf(\mathcal{P}_n) - Tf(\mathcal{P}'_n)\| \xrightarrow{n} 0$. Thus, Tf is Riemann integrable. \square

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